

A GEOMETRICAL APPROACH TO GORDAN–NOETHER’S AND FRANCHETTA’S CONTRIBUTIONS TO A QUESTION POSED BY HESSE.

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ABSTRACT. Hesse claimed in [He1] (and later also in [He2]) that an irreducible projective hypersurface in \mathbb{P}^n defined by an equation with vanishing hessian determinant is necessarily a cone. Gordan and Noether proved in [GN] that this is true for $n \leq 3$ and constructed counterexamples for every $n \geq 4$. Gordan and Noether and Franchetta gave classification of hypersurfaces in \mathbb{P}^4 with vanishing hessian and which are not cones, see [GN] and [Fra]. Here we translate in geometric terms Gordan and Noether approach, providing direct geometrical proofs of these results.

0. INTRODUCTION

Let $f = f(x_0, \dots, x_n) \in k[x_0, \dots, x_n]$ be a non-zero irreducible homogeneous polynomial over an algebraically closed field k of characteristic zero. Then the hessian polynomial of f is the determinant of the matrix of the second derivatives:

$$h_f := \det([\partial^2 f / \partial x_i \partial x_j]_{i,j=0,\dots,n}).$$

Obviously when the hypersurface $X = V(f) \subset \mathbb{P}^n$ is a cone (i.e. up to a linear change of coordinates f does not depend on all the variables) then the hessian polynomial of f is identically zero. The converse is clearly true when $\deg(f) = 2$. Hesse claimed twice that the converse is true for each degree of the polynomial f , i.e. he claimed that if the hessian polynomial of a polynomial f is identically zero then the hypersurface $X = V(f) \subset \mathbb{P}^n$ is a cone (see [He1], [He2]).

The problem was reconsidered by Gordan and Noether ([GN]) who proved that Hesse’s claim is true when $n \leq 3$ but false in general when $n \geq 4$. They constructed families of counterexamples for every $n \geq 4$, which have been revisited recently by Permutti in [Pm1], [Pm2] and by Lossen in [Los]. Moreover, Gordan and Noether seem to have proved that their families of examples are the only possible counterexamples if $n = 4$ but it is rather difficult to indicate a precise reference for this result in their monumental paper. Franchetta ([Fra]) gave an independent classification of hypersurfaces in \mathbb{P}^4 with vanishing hessian which are not cones using more geometrical techniques. Other examples were given by Perazzo, [Per], who considered the case of cubic hypersurfaces with vanishing hessian and obtained the classification of these cubics in \mathbb{P}^4 , \mathbb{P}^5 and \mathbb{P}^6 .

Since the problem posed by Hesse has a geometrical flavour, the aim of this note is to translate in more geometric term Gordan and Noether approach using some ideas and results contained in [GN] and [Los] and in the recent [CRS]. We also briefly describe the counterexamples in projective spaces of dimension at least four produced by Gordan and Noether, relating them to works of Franchetta and Permutti and we will give a short geometrical proof of the characterization of hypersurfaces in \mathbb{P}^4 with vanishing hessian which are not cones.

In the first Section we describe some background materials and we consider a geometrical construction involving the dual variety of a hypersurface. This construction allows us to

reconsider the Gordan and Noether's results and to describe them in a geometrical context. In the second Section the Hesse's claim is proved in the case of hypersurface of dimension at most 2. This proof is very easy and it is based on the geometrical construction given in the first Section. In the third Section the counterexamples by Gordan and Noether and Franchetta are described, using also the results of [Pm1], [Pm2], [CRS]. The last Section is dedicated to hypersurfaces in \mathbb{P}^4 . We describe the properties of hypersurfaces in \mathbb{P}^4 with vanishing hessian and then we give another proof of Franchetta's classification of these hypersurfaces.

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1. BACKGROUND MATERIAL.

1.1. The Polar map and the Hessian of a projective hypersurface. Consider a non-constant homogeneous polynomial of degree $d \geq 1$, $f = f(x_0, \dots, x_n) \in k[x_0, \dots, x_n]$, in the $n + 1$ variables x_0, \dots, x_n over an algebraically closed field k of characteristic zero. Denote by f_i the partial derivatives $\partial f / \partial x_i$, $i = 0, \dots, n$.

Definition 1.1. Let $X = V(f) \subset \mathbb{P}^n$ be the associated hypersurface. We say that X is a *cone* if, modulo projective transformations of \mathbb{P}^n , the equation defining X does not depend on all the variables.

Equivalently X is a cone if and only if $\text{Vert}(X) \neq \emptyset$. The *vertex* of X , $\text{Vert}(X)$, is the set:

$$\text{Vert}(X) := \{x \in X : J(x, X) = X\},$$

where

$$J(x, X) = \overline{\bigcup_{y \neq x, y \in X} \langle x, y \rangle} \subset \mathbb{P}^n$$

is the join of x and X .

We recall that, if $X \subset \mathbb{P}^n$ is an (irreducible) subvariety of $\dim(X) = d$, then

$$\text{Vert}(X) = \bigcap_{x \in X} T_x X = \mathbb{P}^l \subset X,$$

with $l \geq -1$. (see e.g. [Rus], Proposition 1.2.6).

Definition 1.2. The (*first*) *polar map* associated to the hypersurface $X = V(f) \subset \mathbb{P}^n$ is the rational map $\phi_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, defined by the partial derivatives of f :

$$\phi_f(p) = (f_0(p), \dots, f_n(p)).$$

If $p \in X = V(f)$ is not singular, the polar map ϕ_f can be interpreted as mapping the point $p \in X$ to its tangent hyperplane $T_{X,p}$ (and, as such, the target of the map ϕ_f is \mathbb{P}^{n*}). Note that the base locus of ϕ_f is the scheme $\text{Sing}(X) = V(f_0, \dots, f_n) \subset \mathbb{P}^n$. Denote by $Z(f) \subset \mathbb{P}^{n*}$ the closure of the image of \mathbb{P}^n under the polar map ϕ_f . The variety $Z(f) \subset \mathbb{P}^{n*}$ is called the *polar image* of f .

Definition 1.3. We define the *Hessian matrix* of the polynomial f to be the $(n+1) \times (n+1)$ matrix:

$$\mathcal{H}_f := \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j=0,\dots,n}.$$

Its determinant $h_f := \det(\mathcal{H}_f) \in k[x_0, \dots, x_n]$ is the *Hessian polynomial* of f .

Note that the Jacobian matrix J_{ϕ_f} of the (affine) polar map ϕ_f is exactly the Hessian matrix of f , \mathcal{H}_f .

We recall now the construction of the dual variety X^* of an algebraic reduced variety $X \subset \mathbb{P}^n$. Let $\text{Sm}(X)$ denote the open non-empty subset of non singular points of a reduced variety $X \subset \mathbb{P}^n$. Let

$$\mathcal{P}_X := \overline{\{(x, H) : x \in \text{Sm}(X), T_x X \subset H\}} \subset X \times (\mathbb{P}^n)^*$$

be the conormal variety of X , and consider the projections of \mathcal{P}_X onto the factors:

$$\begin{array}{ccc} & \mathcal{P}_X & \\ p_1 \swarrow & & \searrow p_2 \\ X & & X^* = \overline{p_2(\mathcal{P}_X)} \subset (\mathbb{P}^n)^*. \end{array}$$

The dual variety of X , X^* , is the scheme-theoretic image of \mathcal{P}_X in $(\mathbb{P}^n)^*$. In particular if X is a hypersurface of \mathbb{P}^n , then X^* is the closure of the set of hyperplanes tangent to X at non-singular points. Observe that since the Gauss map of X associates to a non singular point $p \in X$ the point in $(\mathbb{P}^n)^*$ corresponding to the hyperplane tangent to X in p , we infer that (when X is a hypersurface) the closure of the image of the Gauss map of X is exactly the dual variety X^* .

Note also that the restriction of the polar map ϕ_f to $V(f) \setminus \text{Sing}(V(f))$ is the Gauss map of $X = V(f)$, hence the closure of the image of X via ϕ_f is the dual variety X^* of X .

1.2. Hypersurfaces with vanishing Hessian. We recall that f_0, \dots, f_n are *algebraically dependent* if there exists a polynomial

$$\pi = \pi(y_0, \dots, y_n) \in k[y_0, \dots, y_n]$$

such that $\pi(f_0, \dots, f_n) = 0$. In particular they are linearly dependent if and only if there exists such a π of degree one.

Note first that the following easy fact holds, recalling that the Jacobian matrix of the affine polar map $\widehat{\phi_f} : k^{n+1} \dashrightarrow k^{n+1}$ is the hessian matrix \mathcal{H}_f .

Proposition 1.4. *Let $f \in k[x_0, \dots, x_n]$ be an homogeneous polynomial, then the following are equivalent:*

- $h_f \equiv 0$;
- ϕ_f is not a dominant map;
- $Z(f) \subsetneq \mathbb{P}^{n*}$;
- f_0, \dots, f_n are algebraically dependent.

We recall the following result from [DP], which proves a conjecture stated in [Dol].

Theorem 1.5. [DP, Corollary 2] *The degree of the polar map ϕ_f depends only on $\text{Supp}(V(f))$ (where the degree of ϕ_f is zero if and only if ϕ_f is not a dominant map).*

Note that by Proposition 1.4 the property of having vanishing Hessian is equivalent to the fact that $\dim(Z(f)) < n$, whence by Theorem 1.5 this property depends only on the support of the hypersurface $X = V(f)$.

Since we are interested in hypersurfaces with vanishing Hessian, from now on we shall assume that $X = V(f)$ is a reduced (and irreducible) hypersurface.

The following result is due to Zak (see [Zak], Proposition 4.9).

Proposition 1.6. *Let $X = V(f) \subset \mathbb{P}^n$ be a reduced hypersurface with vanishing Hessian and let $Z(f) \subsetneq \mathbb{P}^{n*}$ denote the polar image of f . Suppose $d \geq 2$, i.e. ϕ_f not constant. Then:*

$$Z(f)^* \subset \text{Sing}(X).$$

In particular, $\text{Sing}(X) \neq \emptyset$, $\dim(Z(f)^) \leq n - 2$ and $X^* \subsetneq Z(f)$.*

In the sequel we shall need the following well know result, see for example [Ein], Proposition 1.1.

Proposition 1.7. *The hypersurface $V(f) = X$ is a cone if and only if X^* is a degenerate variety. In particular the hypersurface $X = V(f)$ is a cone if and only if the partial derivative of f are linearly dependent.*

Now we recall a problem considered twice by Hesse in [He1] and [He2], giving an equivalent geometric formulation of it. Note that obviously, when $X = V(f) \subset \mathbb{P}^n$ is a cone, i.e. up to a linear change of variables f does not depend on all the variables, then $h_f \equiv 0$. One can ask if the converse holds.

Hesse's problem: If $h_f \equiv 0$, then is $V(f) \subset \mathbb{P}^n$ a cone?

Note that by Proposition 1.7 Hesse's claim is equivalent to prove that if $h_f \equiv 0$ then the derivatives of f are linearly dependent.

The question was reconsidered by Gordan and Noether in [GN]. They showed that Hesse's claim is true when $n \leq 3$ but it is false in general when $n \geq 4$. They constructed families of counterexamples which have been revisited recently by Permutti in [Pm1], [Pm2] and by Lossen in [Los]. An easy example for $n = 4$ is the following cubic polynomial $f(x_0, x_1, x_2, x_3, x_4) = x_0x_3^2 + 2x_1x_3x_4 + x_2x_4^2$.

Remark 1.8. Note that if $d = \deg(f) \leq 2$ then the Hesse's claim is true for every $n \geq 1$. Indeed if $d = 1$ then $V(f)$ is a hyperplane, and so it is a cone. If $d = 2$ then $V(f)$ is a hyperquadric and \mathcal{H}_f is the matrix associated to the quadratic form of $V(f)$. Since its determinant is zero, the associated hyperquadric is singular, and so the hyperquadric is a cone.

From now on $f \in k[x_0, \dots, x_n]$ **is a homogeneous reduced polynomial of degree $d \geq 3$ such that $h_f \equiv 0$.**

Since $h_f \equiv 0$, there exist homogeneous polynomials $\pi \in k[y_0, \dots, y_n]$ such that $\pi(f_0, \dots, f_n) \in k[x_0, \dots, x_n]$ is identically equal to zero. Let $g \in k[y_0, \dots, y_n]$ be such a homogeneous polynomial with this property and such that $g_i := \frac{\partial g}{\partial y_i}(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}) \in k[x_0, \dots, x_n]$, $i = 0, \dots, n$ are not all identically equal to zero.

Definition 1.9. Let $S = V(g) \subset \mathbb{P}^{n*}$ be an irreducible and reduced hypersurface containing the polar image $Z(f)$ and such that $Z(f)$ is not completely contained in the singular locus of S . Let

$$\psi_g: \mathbb{P}^n \dashrightarrow \mathbb{P}^n$$

be the composition of ϕ_f with ϕ_g (or equivalently ψ_g is the composition of ϕ_f with the Gauss map of S). If the polynomials $g_i := \frac{\partial g}{\partial y_i}(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}) \in k[x_0, \dots, x_n]$ have a common divisor $\rho := \text{g.c.d.}(g_0, \dots, g_n) \in k[x_0, \dots, x_n]$, set $h_i := \frac{g_i}{\rho} \in k[x_0, \dots, x_n]$, for $i = 0, \dots, n$.

It follows that the map ψ_g is given by:

$$\psi_g(p) = (g_0(f_0(p), \dots, f_n(p)): \dots : g_n(f_0(p), \dots, f_n(p))) = (h_0(p): \dots : h_n(p)),$$

with $\text{g.c.d.}(h_0, \dots, h_n) = 1$.

So we have:

$$\begin{array}{ccc}
 X & \xrightarrow{\mathcal{G}_X} & X^* \\
 \cap & & \cap \\
 \mathbb{P}^n & \xrightarrow{\phi_f} & Z(f) \subset S = V(g) \subset \mathbb{P}^{n*} \quad \xrightarrow{\phi_g} \quad \mathbb{P}^{n**} \cong \mathbb{P}^n
 \end{array}$$

$$\mathbb{P}^n \quad \xrightarrow{\psi_g = \phi_g \circ \phi_f} \quad \mathbb{P}^{n**} \cong \mathbb{P}^n.$$

Note that the base locus of ψ_g is the scheme $Bs(\psi_g) = V(h_0, \dots, h_n) \subset \mathbb{P}^n$ of codimension at least 2 (because $\text{g.c.d.}(h_0, \dots, h_n) = 1$ and because we can assume that the h_i 's are not constant).

Set $S_Z^* := \overline{\psi_g(\mathbb{P}^n)}$ and note that by definition of ψ_g ,

$$S_Z^* \subset Z(f)^*.$$

Indeed $Z(f)^*$ is made up of the hyperplanes containing the tangent spaces to $Z(f)$, and S_Z^* is made up by the hyperplanes which are tangent to S in the points of $Z(f)$. Since $Z(f) \subset S$ the hyperplanes which are tangent to S in points of $Z(f)$ are clearly hyperplanes containing the tangent spaces to $Z(f)$.

Recalling Proposition 1.6, we get:

$$(1) \quad S_Z^* \subset Z(f)^* \subset \text{Sing}(X).$$

Let us recall a fundamental result proved by Gordan and Noether (see [GN] and also [Los], 2.7).

Theorem 1.10. *Under the above notation, let $F \in k[x_0, \dots, x_n]$. Then:*

$$(2) \quad \sum_{i=0}^n \frac{\partial F}{\partial x_i} h_i = 0 \Leftrightarrow \forall \lambda \in k, F(\underline{x}) = F(\underline{x} + \lambda \psi_g(\underline{x})).$$

Remark 1.11. Note that $\sum_{i=0}^n \frac{\partial^2 f}{\partial x_i \partial x_j} h_i = 0$. This relation is obtained differentiating the equation $g(f_0, \dots, f_n) = 0$ with respect to x_j and applying the chain rule. As a consequence, we get the following relation by Theorem 1.10:

$$(3) \quad f_i(\underline{x}) = f_i(\underline{x} + \lambda \psi_g(\underline{x})).$$

Remark 1.12. Using the above result one can find that: $\sum_{i=0}^n \frac{\partial g_k}{\partial x_i} h_i = 0$.

Indeed since g_k is a polynomial in $\frac{\partial f}{\partial x_j}$, $j = 0, \dots, n$,

$$\sum_{i=0}^n \frac{\partial g_k}{\partial x_i} h_i = \sum_{i=0}^n \left(\sum_{j=0}^n \frac{\partial g_k}{\partial (\frac{\partial f}{\partial x_j})} \cdot \left(\frac{\partial^2 f}{\partial x_j \partial x_i} \right) \right) h_i = 0,$$

where the last equality follows from Remark 1.11.

As a consequence we obtain

$$(4) \quad \sum_{i=0}^n \frac{\partial h_k}{\partial x_i} h_i = \frac{1}{\rho} \sum_{i=0}^n \frac{\partial g_k}{\partial x_i} h_i = 0.$$

Since $\psi_g = (h_0 : \dots : h_n)$, by Theorem 1.10 we have

$$(5) \quad \forall p \in \mathbb{P}^n, \forall \lambda \in k, \psi_g(p) = \psi_g(p + \lambda \psi_g(p)).$$

Geometrically, the equation (5) means that the fiber of ψ_g over a point $q \in S_Z^*$, $\psi_g^{-1}(q)$, is a cone whose vertex contains the point q . Indeed $\forall p \in \mathbb{P}^n$ such that $\psi_g(p) = q$, $q = \psi_g(p) = \psi_g(p + \lambda\psi_g(p)) = \psi_g(p + \lambda q)$, i.e. $p + \lambda q \in \psi_g^{-1}(q)$ for all λ . Hence $\forall p \in \mathbb{P}^n$ such that $\psi_g(p) = q$, the line $\langle p, q \rangle$ is contained in $\psi_g^{-1}(q)$ and $\langle p, q \rangle \cap \text{Bs}(\psi_g) = \{q\}$ as sets.

Remark 1.13. If the condition (2) holds for a polynomial $F \in k[x_0, \dots, x_n]$, then:

$$S_Z^* \subset V(F).$$

Indeed, using the equation (2) and applying Taylor's formula we have:

$$0 = F(\underline{x}) - F(\underline{x} + \lambda\psi_g(\underline{x})) = \sum_{k=1}^e \lambda^k \Phi_k,$$

with $e = \deg(F)$, $\Phi_k := \sum_{i_1 \dots i_k} \frac{\partial^k F}{\partial x_{i_1} \dots \partial x_{i_k}} \frac{h_{i_1} \dots h_{i_k}}{k!}$. In particular, if we assume $F \neq 0$, homogeneous of degree $e \geq 1$, comparing the coefficient for λ^e we get: $F(\psi_g(\underline{x})) = F(h_0, \dots, h_n) = 0$.

Collecting the above remarks we get that the following result.

Proposition 1.14. *Let notation and hypothesis be as above and suppose that $V(f)$ is not a cone. Then:*

$$S_Z^* \subset V(h_0, \dots, h_n) = \text{Bs}(\psi_g).$$

In particular $\dim(S_Z^*) \leq \dim V(h_0, \dots, h_n) \leq n - 2$.

Proof. By the equation (4), $\sum_{i=0}^n \frac{\partial h_k}{\partial x_i} h_i = 0$, hence the condition (2) holds for $\psi_g = (h_0, \dots, h_n)$.

By Remark 1.13 this implies that $S_Z^* \subset V(h_0, \dots, h_n) = \text{Bs}(\psi_g)$.

The bound on the dimension of $V(h_0, \dots, h_n)$ follows from the fact that $\text{g.c.d.}(h_0, \dots, h_n) = 1$ and from the fact that we can suppose that the h_i 's are non-zero and non-constant since $V(f)$ is not a cone. \square

We have the following useful proposition that will be used later.

Proposition 1.15. *Under the above notation, let $q \in S_Z^*$ be a general point and let $w \in \text{Bs}(\psi_g)$ (respectively $t \in \text{Sing}(X)$). If $w \in \overline{\psi_g^{-1}(q)} \setminus \{q\}$, (respectively $t \in \overline{\psi_g^{-1}(q)} \setminus \{q\}$), then the line $\langle w, q \rangle$ is contained in $\text{Bs}(\psi_g)$, (respectively the line $\langle t, q \rangle$ is contained in $\text{Sing}(X)$).*

Proof. Since $\psi_g^{-1}(q)$ is a cone whose vertex contains the point q by (5), then $\overline{\psi_g^{-1}(q)}$ is a cone whose vertex contains the point q . The line $\langle w, q \rangle$, respectively the line $\langle t, q \rangle$, is contained in $\overline{\psi_g^{-1}(q)}$, whence the conclusion follows from the relations (5) and (3). \square

Another general and useful remark is the following lemma which gives a connection between the polar map of the restriction to a hyperplane with the geometry of $Z(f)$ (see [CRS], Lemma 3.10).

Lemma 1.16. *Let $X = V(f) \subset \mathbb{P}^n$ be a hypersurface. Let $H = \mathbb{P}^{n-1}$ be a hyperplane not contained in X and let $h = H^*$ be the corresponding point in \mathbb{P}^{n*} and let π_h denote the projection from the point h . Then:*

$$\phi_{V(f) \cap H} = \pi_h \circ (\phi_{V(f)}|_H).$$

In particular, $Z(V(f) \cap H) \subset \pi_h(Z(f))$, where $Z(V(f) \cap H)$ denotes the closure of the image of the polar map $\phi_{V(f) \cap H}$ of the hypersurface $V(f) \cap H$ of H .

2. CASES IN WHICH HESSE'S CLAIM IS TRUE.

In this section we shall consider some hypotheses under which the conclusion in Hesse's claim holds.

Remark 2.1. i) *Let $S = V(g) \supseteq Z(f)$. If S^* is a cone, then X is a cone. If $Z(f)^*$ is a cone then X is a cone.*

Indeed if S^* (resp. $Z(f)^*$) is a cone, then S (resp. $Z(f)$) is a degenerate variety of $(\mathbb{P}^n)^*$. Since $X^* \subset Z(f) \subset S$, X^* is a degenerate variety, whence X is a cone.

ii) *If $\dim(S^*) = 0$ (resp. $\dim(Z(f)^*) = 0$) then X is a cone.*

By reflexivity we get $S = S^{**} = \mathbb{P}^{n-1}$, resp. $Z(f) = \mathbb{P}^{n-1}$. In both cases the result follows from part i).

Again by part i), if S^* (resp. $Z(f)^*$) is a linear subspace of \mathbb{P}^{n**} then X is a cone (because the dual of linear subspaces of \mathbb{P}^{n**} are linear subspaces of \mathbb{P}^{n*}).

iii) *If $\dim(S_Z^*) = 0$, then X^* is a cone.*

Indeed if $\dim(S_Z^*) = 0$, S_Z^* is a point, and then all the tangent spaces to the points of $Z(f)$ are contained in a hyperplane (the dual of the point S_Z^*). But this means that $Z(f)$ is contained in a hyperplane, whence it is degenerate. It follows that X^* is degenerate and so X is a cone.

In particular we recall some properties of the cone X which are described dually by other geometric properties of its dual variety X^* .

Remark 2.2. i) *If X^* is a non degenerate subvariety of a hyperplane \mathbb{P}^{n-1} in $\mathbb{P}^{n*}(\cong \mathbb{P}^n)$, then X is a cone with vertex exactly a point.*

ii) *if X^* is a non-degenerate subvariety of a linear subspace $L = \mathbb{P}^{n-m}$ ($m = 1, \dots, n-1$) in $\mathbb{P}^{n*}(\cong \mathbb{P}^n)$, then X is a cone with vertex a linear subspace $\mathbb{P}^{m-1} = L^*$.*

iii) *If X^* is union of $d \geq 1$ points which span a linear subspace \mathbb{P}^{n-m} of $(\mathbb{P}^n)^*$, then X is made up by d hyperplanes whose intersection is a $(m-1)$ -linear subspace of \mathbb{P}^n .*

Now we can prove easily Hesse's claim when $n \leq 3$.

Proposition 2.3. *Let $X = V(f) \subset \mathbb{P}^1$ be a reduced hypersurface of degree d . Then $X = V(f)$ has vanishing Hessian if and only if X is a cone. In this case $d = 1$ and X is a point.*

Proof. In this case $Z(f) \subsetneq \mathbb{P}^1$ must be a point because ϕ_f is not dominant, so the partial derivatives of f are constant and $d = 1$ since X is reduced, i.e. X is a point. \square

Proposition 2.4. *Let $X = V(f) \subset \mathbb{P}^2$ be a reduced hypersurface of degree $d \geq 2$. Then $X = V(f)$ has vanishing Hessian if and only if X is a cone, i.e. if and only if X consists of d distinct lines through a point.*

Proof. Note that $\dim(Z(f)) \leq 1$. As in Proposition 2.3, $Z(f)$ is a point if and only if $d = 1$. Assume $\dim(Z(f)) = 1$. By Proposition 1.6, $Z(f)^* \subset \text{Sing}(X)$. Since we are assuming X to be reduced, we infer that $Z(f)^*$ is a point, so $Z(f)$ is a line and whence the hypersurface X is a cone, made up by d lines meeting in the point $Z(f)^*$ (where d is the degree of f). \square

The following result was proved by Gordan and Noether in [GN]. Here we give an easier and more geometrical proof of it.

Proposition 2.5. *Let $X = V(f) \subset \mathbb{P}^3$ be a reduced hypersurface of degree $d \geq 3$. Then $X = V(f)$ has vanishing Hessian if and only if X is a cone. More precisely, $X = V(f)$ has vanishing Hessian if and only if either X is a cone over a curve of vertex a point or X consists of d distinct planes through a line. In the first case $Z(f)$ is a plane in \mathbb{P}^{3*} while in the second case it is a line in \mathbb{P}^{3*} .*

Proof. In this case $\dim(S_Z^*) \leq \dim(Z(f)^*) \leq 1$ by (1) and Proposition 1.6.

Assume $\dim(S_Z^*) = 0$ and we are in the hypothesis of the Remark 2.1 ii), so X is a cone. In particular if $\dim(Z(f)^*) = 0$, X^* is contained in the plane $Z(f) \subset \mathbb{P}^3$. Moreover X^* is non-degenerate in the plane $Z(f)$ because otherwise it would be either a line or a point, which is clearly impossible. It follows from Remark 2.2 that X is a cone with vertex the point $Z(f)^*$ over a plane curve (the dual curve of X^* in the plane $Z(f)$).

Assume now that $\dim(S_Z^*) = 1$. Since $\dim(Z(f)^*) \leq 1$ and $S_Z^* \subset Z(f)^*$, this implies that $\dim(Z(f)^*) = 1$. Since $Z(f)^*$ and S_Z^* are irreducible (because $Z(f)$ is irreducible), $Z(f)^* = S_Z^*$.

Let s_1, s_2 two distinct general point of S_Z^* . Then $\overline{\psi_g^{-1}(s_i)}$ is a surface which is a cone whose vertex contains the point s_i . Let $t \in \psi_g^{-1}(s_1) \cap \psi_g^{-1}(s_2) \subset \text{Bs}(\psi_g)$. By Proposition 1.15, the lines $\langle s_i, t \rangle$, $i = 1, 2$, are contained in the base locus of ψ_g . Since $\dim \text{Bs}(\psi_g) \leq 1$, the irreducible component of $\text{Bs}(\psi_g)$ passing through s_1 is exactly the line $\langle s_1, t \rangle$. But also S_Z^* is an irreducible component of $\text{Bs}(\psi_g)$ of dimension one passing through s_1 , so it has to coincide with the line $\langle s_1, t \rangle$. We conclude that $S_Z^* = Z(f)^* = \langle s_i, t \rangle = \langle s_1, s_2 \rangle$.

Since $Z(f)^*$ is a line, then $Z(f)$ is a line and $X^* \subsetneq Z(f) = \mathbb{P}^1$, whence X is the union of d planes through $Z(f)^* = \mathbb{P}^1$ by Remark 2.2. \square

Corollary 2.6. *Let $X = V(f) \subset \mathbb{P}^n$, $n \geq 4$ be a reduced hypersurface of degree d . If $X = V(f)$ has vanishing Hessian and if $\dim(Z(f)) \leq 2$, then $X = V(f)$ is a cone.*

Proof. Let $H \subset \mathbb{P}^n$ be a general \mathbb{P}^3 and let $h = H^* = \mathbb{P}^{n-4}$. By iterating Lemma 1.16 we deduce that the variety $Z(V(f) \cap H)$ is contained in the variety $\pi_h(Z(f))$, whose dimension equals $\dim(Z(f))$. Thus $V(f) \cap H$ has vanishing Hessian because the polar map $\phi_{V(f) \cap H}: \mathbb{P}^3 \dashrightarrow \mathbb{P}^{3*}$ is not dominant. By Proposition 2.5 we infer that $V(f) \cap H$ is a cone. By the generality of H we get that $X = V(f) \subset \mathbb{P}^n$ is a cone. \square

3. GORDAN–NOETHER AND FRANCHETTA COUNTEREXAMPLES TO HESSE’S CONJECTURE.

In this section we will describe some examples of hypersurfaces in \mathbb{P}^n , $n \geq 4$, with vanishing Hessian and which are not cones, following [GN] and [CRS, §2]. Moreover we introduce the hypersurfaces in \mathbb{P}^4 which are counterexamples to Hesse’s claim described by Franchetta (cf. [Fra]). We observe that these hypersurfaces are particular cases of the ones described by Gordan–Noether.

We briefly recall the results of Gordan–Noether and Permutti in connection with the Hesse problem, following [CRS].

Assume $n \geq 4$ and fix integers $t \geq m + 1$ such that $2 \leq t \leq n - 2$ and $1 \leq m \leq n - t - 1$. Consider forms $h_i(y_0, \dots, y_m) \in k[y_0, \dots, y_m]$, $i = 0, \dots, t$, of the same degree, and also forms $\psi_j(x_{t+1}, \dots, x_n) \in k[x_{t+1}, \dots, x_n]$, $j = 0, \dots, m$, of the same degree. Introduce the following homogeneous polynomials all of the same degree:

$$Q_\ell(x_0, \dots, x_n) := \det \begin{pmatrix} x_0 & \dots & x_t \\ \frac{\partial h_0}{\partial \psi_0} & \dots & \frac{\partial h_t}{\partial \psi_0} \\ \dots & \dots & \dots \\ \frac{\partial h_0}{\partial \psi_m} & \dots & \frac{\partial h_t}{\partial \psi_m} \\ a_{1,0}^{(\ell)} & \dots & a_{1,t}^{(\ell)} \\ \dots & \dots & \dots \\ a_{t-m-1,0}^{(\ell)} & \dots & a_{t-m-1,t}^{(\ell)} \end{pmatrix}$$

where $\ell = 1, \dots, t-m$. Note that $a_{u,v}^{(\ell)} \in k$ for $u = 1, \dots, t-m-1$, $v = 0, \dots, t$ and $\frac{\partial h_i}{\partial \psi_j}$ stands for the derivative $\frac{\partial h_i}{\partial y_j}$ computed at $y_j = \psi_j(x_{t+1}, \dots, x_n)$ for $i = 0, \dots, t$ and $j = 0, \dots, m$. Let s denote the common degree of the polynomials Q_ℓ . Taking Laplace expansion along the first row, one has an expression of the form:

$$Q_\ell = M_{\ell,0}x_0 + \dots + M_{\ell,t}x_t,$$

where $M_{\ell,i}$, $\ell = 1, \dots, t-m$, $i = 0, \dots, t$ are homogeneous polynomials of degree $s-1$ in x_{t+1}, \dots, x_n .

Fix an integer $d > s$ and set $\mu = \lfloor \frac{d}{s} \rfloor$. Fix biforms $P_k(z_1, \dots, z_{t-m}; x_{t+1}, \dots, x_n)$ of bidegree $k, d-ks$, $k = 0, \dots, \mu$. Finally set

$$(6) \quad f(x_0, \dots, x_n) := \sum_{k=0}^{\mu} P_k(Q_1, \dots, Q_{t-m}, x_{t+1}, \dots, x_n),$$

a form of degree d in x_0, \dots, x_n . The polynomial f is called a *Gordan–Noether polynomial* (or a *GN–polynomial*) of type (n, t, m, s) , and so will also any polynomial which can be obtained from it by a projective change of coordinates. Accordingly, a *Gordan–Noether hypersurface* (or a *GN–hypersurface*) of type (n, t, m, s) is the hypersurface $V(f)$, where f is a GN–polynomial of type (n, t, m, s) .

The main point of the Gordan–Noether construction is that a GN–polynomial has vanishing Hessian. For a proof see [CRS, Proposition 2.9]. Another proof closer to Gordan–Noether’s original approach is contained in [Los].

Proposition 3.1. *Every GN–polynomial has vanishing Hessian.*

Following [Pm2] and [CRS] we give a geometric description of a GN–hypersurface of type (n, t, m, s) as follows. The main result is that the GN–hypersurfaces have vanishing Hessian (cf. 3.1) but in general they are not cones, so they are counterexample to Hesse’s conjecture.

Definition 3.2. Let f be a GN–hypersurface of type (n, t, m, s) . The *core* of $V(f)$ is the t -dimensional subspace $\Pi \subset V(f)$ defined by the equations $x_{t+1} = \dots = x_n = 0$.

We will call a GN–hypersurface of type (n, t, m, s) *general* if the defining data, namely the polynomials $h_i(y_0, \dots, y_m)$, $i = 0, \dots, t$, the polynomials $\psi_j(x_{t+1}, \dots, x_n)$, $j = 0, \dots, m$ and the constants $a_{u,v}^{(\ell)}$, $\ell = 1, \dots, t-m$, $u = 1, \dots, t-m-1$, $v = 0, \dots, t$, have been chosen generically.

Proposition 3.3. ([CRS, Proposition 2.11]) *Let $V(f) \subset \mathbb{P}^n$ be a GN–hypersurface of type (n, t, m, s) and degree d . Set $\mu = \lfloor \frac{d}{s} \rfloor$. Then*

- i) $V(f)$ has multiplicity $d - \mu$ at the general point of its core Π .
- ii) The general $(t+1)$ -dimensional subspace $\Pi_\xi \subset \mathbb{P}^n$ through Π cuts out on $V(f)$, off Π , a cone of degree μ whose vertex is a m -dimensional subspace $\Gamma_\xi \subset \Pi$.
- iii) As Π_ξ varies the corresponding subspace Γ_ξ describes the family of tangent spaces to an m -dimensional unirational subvariety $S(f)$ of Π .
- iv) If $V(f)$ is general and $\mu > n - t - 2$ then $V(f)$ is not a cone.
- v) The general GN–hypersurface is irreducible.

Definition 3.4. ([Fra]) A reduced hypersurface $F = V(f) \subset \mathbb{P}^4$ of degree d is said to be a *Franchetta hypersurface* if it is swept out by a one-dimensional family Σ of planes such that:

- all the planes of the family Σ are tangent to a plane rational curve C (of degree $p > 1$) lying on F ;

- the family Σ and the curve C are such that for a general hyperplane $H = \mathbb{P}^3 \subset \mathbb{P}^4$ passing through C , the intersection $H \cap F$, off the linear span of C , is the union of planes of Σ all tangent to the curve C in the same point p_H . \square

Remark 3.5. Note that by Proposition 3.3 a GN-hypersurface $X = V(f) \subset \mathbb{P}^4$ of type $(4, 2, 1, s)$ is a Franchetta hypersurface with core the linear span of the curve C . On the contrary Permutti proved in [Pm1] that a Franchetta hypersurface $V(f) \subset \mathbb{P}^4$ is a GN-hypersurface of type $(4, 2, 1, s)$. In particular (by Proposition 3.1) a Franchetta hypersurface has vanishing Hessian. This fact can be proved directly see also [Pm1] and [CRS, Proposition 2.18].

4. A GEOMETRICAL PROOF OF GORDAN–NOETHER AND FRANCHETTA CLASSIFICATION OF HYPERSURFACES IN \mathbb{P}^4 WITH VANISHING HESSIAN.

In the previous section we saw that the classes of GN-hypersurfaces of type $(4, 2, 1, s)$ and of Franchetta hypersurfaces coincide. In this section we use the geometrical methods developed in the first section and some other easy facts to provide a short and selfcontained proof of Franchetta characterization of hypersurfaces with vanishing Hessian in \mathbb{P}^4 , [Fra]. So we will prove in a geometrical way that the hypersurfaces in \mathbb{P}^4 with vanishing Hessian are either cones or Franchetta hypersurfaces and that there are no other possibilities. A similar result is not known in higher dimension.

First we give a preliminary result describing a geometrical consequence of the vanishing of the hessian of hypersurfaces in \mathbb{P}^4 , not cones.

Proposition 4.1. *Let $X = V(f) \subset \mathbb{P}^4$ be a reduced hypersurface of degree $d \geq 3$, not a cone. If $X = V(f)$ has vanishing Hessian then $Z(f)^* \subset \mathbb{P}^4$ is an irreducible plane rational curve. Equivalently $Z(f)$ is a cone with vertex a line over an irreducible plane rational curve.*

Proof. By Corollary 2.6, we can suppose $\dim(Z(f)) \geq 3$. Thus $Z(f)^* = S_Z^*$, and $Z(f) = V(g)$ with $g \in k[y_0, \dots, y_4]$ an irreducible polynomial. Note that by Proposition 1.14 we have $1 \leq \dim(Z(f)^*) \leq \dim(\text{Bs}(\psi_g)) \leq 2$.

Assume first $\dim(Z(f)^*) = 2$ so that $Z(f)^*$ is an irreducible component of $\text{Bs}(\psi_g)$. Consider the intersection between the closure of the fibers on two different general points, $s_1, s_2 \in Z(f)^*$. The fiber on each of these points has dimension two, so there exists $t \in \psi_g^{-1}(s_1) \cap \psi_g^{-1}(s_2)$. By Proposition 1.15, the lines $\langle s_i, t \rangle$, $i = 1, 2$, are contained in $\text{Bs}(\psi_g)$ and hence in the irreducible component of it containing s_1 and s_2 . Since s_1 and s_2 are general points in $Z(f)^*$, $Z(f)^*$ is the unique irreducible component of $\text{Bs}(\psi_g)$ containing them. Furthermore $Z(f)^*$ is a ruled surface (because through a general point $s \in Z(f)^*$ there passes a line ℓ_s contained in $Z(f)^*$), which is a cone (because $\ell_{s_1} \cap \ell_{s_2} \neq \emptyset$ for $s_1, s_2 \in Z(f)^*$ general points), whence by Remark 2.1, X is a cone.

Thus we can assume $\dim(Z(f)^*) = 1$. Let s_1 and s_2 be two general points of $Z(f)^*$. Then the intersection $\overline{\psi_g^{-1}(s_1)} \cap \overline{\psi_g^{-1}(s_2)}$ is a surface, say R , contained in $\text{Bs}(\psi_g)$. Note that this intersection has to stabilize for general points of $Z(f)^*$.

Furthermore for every point $t \in R$ and for a general point $s \in Z(f)^*$, by Proposition 1.15, the line $\langle s, t \rangle$ is contained in $\text{Bs}(\psi_g) \cap \overline{\psi_g^{-1}(s)}$, and hence in R . It follows that $Z(f)^*$ is contained in the vertex of the surface R , and that R (and in fact the intersection of two general fibers of ψ_g) is a plane ($Z(f)^*$ is not a line by assumption, so it cannot be contained in the intersection of two or more planes).

In other words $Z(f)^*$ is a plane curve, whose linear span $\Pi = R$ is an irreducible component

of $\text{Bs}(\psi_g)$. Since $S_Z^* = Z(f)^*$, Proposition 1.15 and the same argument used above show that $Z(f)^*$ is contained in the vertex of $\text{Sing}(X)$. Thus the irreducible components of $\text{Sing}(X)$ of dimension 2 are planes containing $Z(f)^*$ so that there is a unique irreducible component of $\text{Sing}(X)$ which is a plane: the linear span of $Z(f)^*$, i.e. Π .

Note also that $Z(f)^*$ is rational. In fact the map ψ_g is a rational dominant map from \mathbb{P}^4 to $Z(f)^*$, so $Z(f)^*$ is a unirational curve and hence a rational curve.

Since $Z(f)^* = S_Z^* \subset \Pi = \mathbb{P}^2$ is an irreducible rational plane curve (not a line), $Z(f)$ is a cone of vertex a line $L = \Pi^* = \mathbb{P}^1$ over an irreducible plane curve Γ (of degree ≥ 2), which is the dual curve of $Z(f)^*$ in the plane Π . Furthermore Γ is a rational curve because in this case the Gauss map of the curve $Z(f)^*$ is birational. \square

The description given in Proposition 4.1 is crucial to prove that a projective hypersurface $X = V(f)$ in \mathbb{P}^4 with vanishing Hessian which is not a cone is a Franchetta hypersurface. The following result finally gives a characterization of hypersurfaces in \mathbb{P}^4 with vanishing Hessian, which are not cones.

Theorem 4.2. *Let $X = V(f) \subset \mathbb{P}^4$ be an irreducible and reduced hypersurface of degree $d \geq 3$, not a cone. The following conditions are equivalent:*

- i) $X = V(f)$ has vanishing Hessian.
- ii) $X = V(f)$ is a Franchetta hypersurface.
- iii) $X^* = V(f)^*$ is a scroll surface of degree d , having a line directrix L of multiplicity e , sitting in a 3-dimensional rational cone $W(f)$ with vertex L , and the general plane ruling of the cone cuts $V(f)^*$ off L along $\mu \leq e$ lines of the scroll, all passing through the same point of L .
- iv) $X = V(f)$ is a general GN-hypersurface of type $(4, 2, 1, s)$, with $\mu = [\frac{d}{s}]$, which has a plane of multiplicity $d - \mu$.

In particular, $X^ = V(f)^*$ is smooth if and only if $d = 3$, $X^* = V(f)^*$ is a rational normal scroll of degree 3 and $X = V(f)$ contains a plane, the orthogonal of the line directrix of $X^* = V(f)^*$, with multiplicity 2.*

Proof. Note that conditions ii) and iii) are easily seen to be equivalent (the directrix line L of X^* is the dual of the plane which is the linear span of the curve C of the Franchetta hypersurface). By Remark 3.5, the equivalence of ii) and iv) is clear. The conditions iv) implies the condition i) by Proposition 3.1. Thus to finish the proof it is sufficient to prove that a hypersurface $X = V(f) \subset \mathbb{P}^4$ with vanishing Hessian, not a cone, is a Franchetta hypersurface.

By Proposition 4.1, we have that $Z(f)^* \subset \text{Sing}(X) \subset X = V(f)$ is an irreducible plane rational curve, whose linear span is a plane $\Pi = \mathbb{P}^2$. Equivalently, $Z(f)$ is a cone of vertex the line $L = \Pi^* = \mathbb{P}^1$ over an irreducible plane curve Γ , the dual of $Z(f)^*$ as a plane curve. Consider now a general hyperplane $H \subset \mathbb{P}^4$ passing through the plane Π (and not contained in $X = V(f)$). The intersection $X \cap H$ is a hypersurface in $H = \mathbb{P}^3$ containing the plane Π with a certain multiplicity $\mu \geq 0$ and reduced elsewhere. Note also that the point $h = H^* \in L = \Pi^*$ (because $\Pi \subset H$), whence $\pi_h(Z(f))$ is a surface naturally embedded in the dual space of H . More precisely $\pi_h(Z(f))$ is a cone with vertex the point $p_L = \pi_h(L)$ over the plane curve $\hat{\Gamma} = \pi_h(\Gamma)$.

By Lemma 1.16 we infer that $Z(V(f) \cap H) \subset \pi_h(Z(f)) \subset \mathbb{P}^{3*}$, whence (see Proposition 1.4) the hypersurface $V(f) \cap H \subset H = \mathbb{P}^3$ has vanishing Hessian. By Proposition 2.5 it follows that $V(f) \cap H$ is a cone and either it is a cone over a curve of vertex a point or $V(f) \cap H$ consists of distinct planes passing through a line. In the first case $Z(V(f) \cap H)$ is a plane in \mathbb{P}^{3*} but

this is not possible because the cone $\pi_h(Z(f))$ is non degenerate. Therefore $Z(V(f) \cap H)$ is a line in H^* and $V(f) \cap H$ is a union of planes through the line $T = Z(V(f) \cap H)^* \subset H$, where duality is considered between H and H^* . Since the hyperplane section $V(f) \cap H$ is singular and since H was general through Π , we deduce that $L = \Pi^* \subset X^*$.

Note that, by Lemma 1.16, $Z(V(f) \cap H) = \phi_{V(f) \cap H}(H) = \pi_h(\phi_f(H))$ is a line contained in $\pi_h(Z(f))$, whence $\phi_f(H)$ is a plane of the ruling of $Z(f)$ corresponding to a point $y \in \Gamma$. Furthermore the lines $L_j := \Pi_j^*$, duals to the planes in $V(f) \cap H$ different from Π , pass all through the point $h = H^*$ and are contained in the plane $T^* = \phi_f(H)$ and in X^* .

Let $z = \psi_g(H) \in Z(f)^*$. Then $\phi_f(H)^* = T_z(Z(f)^*) = T$, i.e. the line of intersection of the planes in $V(f) \cap H$ is the tangent line to the plane curve $Z(f)^*$ in the point z . In conclusion $X = V(f) \subset \mathbb{P}^4$ is a Franchetta hypersurface, where we can take as the one dimensional family Σ of planes contained in X exactly the intersection of a general \mathbb{P}^3 through Π with $X = V(f)$ (i.e. the intersection of the fibers of ψ_g with $X = V(f)$) and we consider as the curve C (cf. Definition 3.4) the curve $Z(f)^*$. \square

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